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CHARACTERISTICS OF LOCAL COMPLIANCE OF AN ELASTIC BODY UNDER A SMALL PUNCH INDENTED INTO THE PLANE PART OF ITS BOUNDARY

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An asymptotic solution of the contact problem of an elastic body indented (without friction) by a circular punch with a flat base is obtained under the assumption of a small relative size of the contact zone. The resulting formulas involve integral characteristics of the elastic body, which depend on its shape, dimensions, fixing conditions, Poisson's ratio, and location of the punch center. These quantities have the mechanical meaning of the coefficients of local compliance of the elastic body. Relations that, generally, reduce the number of independent coefficients in the asymptotic expansion are obtained on the basis of the reciprocal theorem. Some coefficients of local compliance at the center of an elastic hemisphere are calculated numerically. The asymptotic model of an elastic body loaded by a point force is discussed.

Introduction. The need for more accurate calculations of contact pressures between machine elements stimulates the studies of contact problems of the theory of elasticity for bodies different from a half-space. The existing solutions for an elastic layer [1, 2], elastic spatial wedge [3], and elastic truncated sphere [4, 5] are based on the explicit representation of the Green function. Hetényi [6] and Sheveleva [7] developed a method of specular reflection for constructing approximately the Green functions for elastic spatial quarter and octant. Tsvetkov and Chebakov [8] and Aleksandrov and Pozharskii [9] developed a homogeneous-solution method for an elastic plate. Many numerical algorithms for solving contact problems have been proposed (see, e.g., [10–13]).

Under the assumption of a small contact zone, contact problems can be solved by asymptotic methods [1–3, 5]. Argatov [14] showed that, to construct several first terms of the asymptotic expansion of the contact-pressure intensity, it suffices to know several coefficients of the asymptotic expansions of regular components of singular solutions with singularities corresponding to the point force (Green function), point moments, and polymoments. In the present paper, these coefficients are interpreted as characteristics of local compliance of an elastic body.

1. Formulation of the Contact Problem for a Punch with a Flat Base. Local-Compliance Matrix. We consider an elastic body occupying a three-dimensional domain Ω. At the boundary of the body, there is a site Σ, which lies in the Ox_1x_2 plane. Let a circular punch whose center coincides with the coordinate origin be pressed frictionlessly into Σ. We assume that the radius a_ε of the punch base $\omega(\varepsilon)$ is small compared to the characteristic dimension l of the body Ω. For convenience, we set

$$a_\varepsilon = \varepsilon a^*, \tag{1.1}$$

where ε is a small positive parameter and a^* is a quantity comparable with l and independent of ε. As l, we take the radius of the largest sphere with a center at the point O that can be enclosed in the region Ω. Let the body be fixed along the part of the boundary Γ_u and stress-free on Γ_σ and Σ outside the contact region.

We denote the resultant vector and resultant moments of the system of forces acting on the punch by F_3 and M_1 and M_2, respectively. As a result of loading, the punch moves translationally for the distance δ_0 and rotates.

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The rotation is determined by the angles β_1 and β_2. The contact-pressure intensity under the punch p satisfies the following integral equation (see, e.g., [1, § 19]):

$$\iint\limits_{\omega(\varepsilon)} G_3(y_1, y_2; x_1, x_2, 0)\, p(y_1, y_2)\, dy_1\, dy_2 = \delta_0 + \beta_1 x_2 - \beta_2 x_1. \tag{1.2}$$

Here G_3 is the vertical component of the Green vector function with a pole at the boundary point $(y_1, y_2, 0)$.

The unknown quantities δ_0, β_1, and β_2 are determined from the equations of equilibrium of the punch

$$\iint\limits_{\omega(\varepsilon)} p(\boldsymbol{y})\, d\boldsymbol{y} = F_3, \qquad \iint\limits_{\omega(\varepsilon)} \begin{Bmatrix} y_2 \\ -y_1 \end{Bmatrix} p(\boldsymbol{y})\, d\boldsymbol{y} = \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}. \tag{1.3}$$

Remark 1. The contact pressure under the punch base should be positive. Skewness of the punch may lead to detachment of the punch edge from the surface of the elastic body. However, it follows from the solution of the axisymmetric contact problem for an elastic truncated sphere (see [5, § 5.2.1]) that, in the case of translational indentation of the punch, the condition of full contact holds for reasonably small values of the parameter ε only.

Since the problem is linear, the quantities δ_0, β_1, and β_2 should be related to F_3, M_1, and M_2 by the linear relation

$$\begin{pmatrix} \Pi_{00} & \Pi_{01} & \Pi_{02} \\ \Pi_{10} & \Pi_{11} & \Pi_{12} \\ \Pi_{20} & \Pi_{21} & \Pi_{22} \end{pmatrix} \begin{pmatrix} F_3 \\ M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} \delta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}. \tag{1.4}$$

The quantities $\Pi_{kl}(\varepsilon)$ characterize the compliance of the elastic body Ω under the action of the punch with a flat base $\omega(\varepsilon)$ applied at the point O at the face of the body Σ. For small ε, the displacement of the punch is mainly due to the local deformation of the body Ω that occurs in the neighborhood of the punch (according to terminology of [15, § 133], in the local-perturbation region). We call matrix (1.4) the local-compliance matrix. By virtue of the reciprocal theorem $\delta_0'' F_3' + \beta_1'' M_1' + \beta_2'' M_2' = \delta_0' F_3'' + \beta_1' M_1'' + \beta_2' M_2''$, the local-compliance matrix Π is symmetric.

One of the goals of the present paper is to construct an asymptotic representation of the matrix $\Pi(\varepsilon)$ as $\varepsilon \to 0$.

2. Asymptotic Modeling of Local Contact Interaction between an Elastic Body and a Punch. We consider the expansion (see, e.g., [1; 16, § 4.14])

$$G_3(\boldsymbol{y}; \boldsymbol{x}) = T_3(x_1 - y_1, x_2 - y_2, x_3) + g_3(\boldsymbol{y}; \boldsymbol{x}), \tag{2.1}$$

where $g_3(\boldsymbol{y}; \boldsymbol{x})$ is the projection of the regular component of the Green vector function onto the vertical axis, T_3 is the part of the solution of the Boussinesq problem of an elastic half-space loaded by a unit force (see [16, § 5.11]), where

$$\frac{\pi E}{1 - \nu^2}\, T_3(x_1 - y_1, x_2 - y_2, 0) = \frac{1}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}}. \tag{2.2}$$

Here E and ν are Young's modulus and Poisson's ratio of the material of the body Ω, respectively.

We use the following asymptotic formula (notation coincides with that in [14, 17]):

$$\begin{aligned} (\pi E/(1 - \nu^2)) g_3(\boldsymbol{y}; x_1, x_2, 0) &= A_0 + B_1 x_1 + B_2 x_2 - A_0^{(2)} y_1 + A_0^{(1)} y_2 \\ &+ C_{11} x_1^2 + 2C_{12} x_1 x_2 + C_{22} x_2^2 - (B_1^{(2)} x_1 + B_2^{(2)} x_2) y_1 \\ &+ (B_1^{(1)} x_1 + B_2^{(1)} x_2) y_2 + (1/2)(A_0^{(2,0)} y_1^2 + 2A_0^{(2,1)} y_1 y_2 + A_0^{(2,2)} y_2^2) + \dots\ . \end{aligned} \tag{2.3}$$

Here the dots denote the terms of order $O(\varepsilon^3)$ [in accordance with (1.1)]. It is noteworthy that, generally, the coefficients that enter the right side of (2.3) (the method of calculating these coefficients is given in [14, 17]) depend on the location of the point O (punch center).

Example 1. Let Ω be a layer of thickness h perfectly attached to the rigid base $x_3 = h$. In this case, the right side of (2.3) depends only on the squared distance between the points (y_1, y_2) and (x_1, x_2). According to [1], the following representation in the form of an absolutely convergent power series is valid for $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < 2h$:

$$\frac{\pi E}{1 - \nu^2}\, g_3(\boldsymbol{y}; x_1, x_2, 0) = -\frac{1}{h} \sum_{m=0}^{\infty} \frac{a_m}{h^{2m}} [(x_1 - y_1)^2 + (x_2 - y_2)^2]^m.$$

Comparing this expansion with (2.3), we infer that the only nonzero coefficients are

$$A_0 = -a_0/h, \qquad C_{11} = C_{22} = -a_1/h^3,$$
$$A_0^{(2,0)} = A_0^{(2,2)} = -2a_1/h^3, \qquad B_1^{(2)} = -2a_1/h^3, \qquad B_2^{(1)} = 2a_1/h^3. \tag{2.4}$$

The dimensionless coefficients a_0 and a_1 as functions of ν are given in [5, Table 1.2] (see also [18]). For example, $a_0 = 1.3769$, and $a_1 = -0.6276$ for $\nu = 0.3$.

Substituting (2.1)–(2.3) into (1.2) and integrating, we obtain the equation

$$\frac{1-\nu^2}{\pi E} \iint\limits_{\omega(\varepsilon)} \frac{p(y_1, y_2)\, dy_1\, dy_2}{\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}} = \delta_0 + \beta_1 x_2 - \beta_2 x_1 - \tilde{F}_3 A_0 - \tilde{F}_3 (B_1 x_1 + B_2 x_2)$$
$$- \sum_{i=1}^{2} \tilde{M}_i A_0^{(i)} - \tilde{F}_3 (C_{11} x_1^2 + 2C_{12} x_1 x_2 + C_{22} x_2^2) - \sum_{i=1}^{2} \tilde{M}_i (B_1^{(i)} x_1 + B_2^{(i)} x_2) - \sum_{n=0}^{2} \tilde{M}_{2,n} A_0^{(2,n)}. \tag{2.5}$$

Here $\tilde{F}_3$ and $\tilde{M}_1$, and $\tilde{M}_2$ are the integral characteristics of the contact-pressure intensity (normalized force and moments, respectively), which are equal to the quantities calculated from formulas (1.3) multiplied by $(\pi E)^{-1}(1-\nu^2)$ and $\tilde{M}_{2,n}$ are the normalized polymoments of the distributed contact pressures:

$$\tilde{M}_{2,n} = \frac{1-\nu^2}{\pi E} M_{2,n}, \qquad M_{m,n} = \frac{1}{2} C_2^n \iint\limits_{\omega(\varepsilon)} y_1^{2-n} y_2^n p(\boldsymbol{y})\, d\boldsymbol{y}. \tag{2.6}$$

Equation (2.5) is a so-called "coupled" integral equation of the contact problem for a finite elastic body [14] and is a third-approximation equation: the coefficient A_0 is a first-order correction for the geometry of the elastic body, the fifth and sixth terms are second-order corrections, and the next terms are third-order corrections.

The method of reducing the integral equation (1.2) of the contact problem for an elastic layer to an approximate equation by polynomial approximation of the regular component of the integral-operator kernel was proposed in [19] (see also [1, § 54]). The properties of the solutions of these equations are discussed in [5, § 1.2]. The term "coupled" means that, after a solution (with undetermined coefficients) of the integral equation of the contact problem for an elastic half-space is constructed, Eq. (2.5) reduces to a system of linear algebraic equations. Aleksandrov and Shmatkova [18] applied this method to the problem of a parabolic punch pressed into an elastic layer. Argatov [17] obtained an asymptotic solution of the corresponding nonlinear resulting problem.

3. Solution of the Coupled Equation. Using the results of [20–22], we write the solution of Eq. (2.5) in the form

$$p(x_1, x_2) = \frac{E}{\pi(1-\nu^2)} \frac{1}{\sqrt{a^2 - x_1^2 - x_2^2}} \Big[\delta_0 - \tilde{F}_3 A_0 - \sum_{i=1}^{2} \tilde{M}_i A_0^{(i)} - \sum_{n=0}^{2} \tilde{M}_{2,n} A_0^{(2,n)}$$
$$+ \tilde{F}_3 (C_{11} + C_{22}) a^2 - 2\Big(\beta_2 + \tilde{F}_3 B_1 + \sum_{i=1}^{2} \tilde{M}_i B_1^{(i)}\Big) x_1 + 2\Big(\beta_1 - \tilde{F}_3 B_2 - \sum_{i=1}^{2} \tilde{M}_i B_2^{(i)}\Big) x_2$$
$$- \frac{2}{3} \tilde{F}_3 (5C_{11} + C_{22}) x_1^2 - \frac{16}{3} \tilde{F}_3 C_{12} x_1 x_2 - \frac{2}{3} \tilde{F}_3 (C_{11} + 5C_{22}) x_2^2\Big]. \tag{3.1}$$

The quantities $\tilde{F}_3$, $\tilde{M}_i$, and $\tilde{M}_{2,n}$ should be related to δ_0, β_1, and β_2. Integration of (3.1) yields

$$\tilde{F}_3 = c\Big[\delta_0 - \tilde{F}_3 A_0 - \sum_{i=1}^{2} \tilde{M}_i A_0^{(i)} - \sum_{n=0}^{2} \tilde{M}_{2,n} A_0^{(2,n)} - \frac{1}{3} (C_{11} + C_{22}) a^2 \tilde{F}_3\Big], \tag{3.2}$$

where $c = 2\pi^{-1} a$ is the translational capacity of the circular punch of base radius a (dependence on the parameter ε is not indicated).

Calculating the moments of the contact-pressure intensity (3.1) [see the second formula in (1.3)], we obtain

$$\tilde{M}_1 = m\Big(\beta_1 - \tilde{F}_3 B_2 - \sum_{i=1}^{2} \tilde{M}_i B_2^{(i)}\Big), \quad \tilde{M}_2 = m\Big(\beta_2 + \tilde{F}_3 B_1 + \sum_{i=1}^{2} \tilde{M}_i B_1^{(i)}\Big), \tag{3.3}$$

where $m = 4(3\pi)^{-1}a^3$ is the rotational capacity of the circular punch.

Finally, inserting (3.1) into (2.6), we find that

$$\tilde{M}_{2,0} = \frac{a^3}{3\pi}\Big(\delta_0 + \tilde{F}_3\Big[A_0 - \frac{a^2}{15}\,(17C_{11} + C_{22})\Big] - \sum_{i=1}^{2} \tilde{M}_i A_0^{(i)} - \sum_{n=0}^{2} \tilde{M}_{2,n} A_0^{(2,n)}\Big); \tag{3.4}$$

$$\tilde{M}_{2,1} = -\frac{32a^5}{45\pi}\,\tilde{F}_3 C_{12}; \tag{3.5}$$

$$\tilde{M}_{2,2} = \frac{a^3}{3\pi}\Big(\delta_0 + \tilde{F}_3\Big[A_0 - \frac{a^2}{15}\,(C_{11} + 17C_{22})\Big] - \sum_{i=1}^{2} \tilde{M}_i A_0^{(i)} - \sum_{n=0}^{2} \tilde{M}_{2,n} A_0^{(2,n)}\Big). \tag{3.6}$$

With allowance for (3.5), one can express the quantities $\tilde{M}_{2,0}$ and $\tilde{M}_{2,2}$ from Eqs. (3.4) and (3.6). The determinant of this system is equal to $1+(3\pi)^{-1}a^3(A_0^{(2,0)}+A_0^{(2,2)})$ and does not vanish for reasonably small $\varepsilon = a/l$. However, the exact solution is not necessarily required, since formulas (3.1)–(3.6) themselves are approximate.

4. Asymptotic Representation of the Local-Compliance Matrix. We now perform an asymptotic analysis of relations (3.4)–(3.6) (see also [17]). Let the quantities δ_0, β_1, and β_2 be fixed (independent of the parameter ε). Then, from (3.2) follows the expansion $\tilde{F}_3 = \varepsilon \tilde{F}_3^0 + \varepsilon^2 \tilde{F}_3^1 + \ldots$. It should be borne in mind that formula (3.1) is derived by retaining terms of order ε^3 compared to unity [see (2.5), where $\sqrt{x_1^2 + x_2^2} < \varepsilon a^*$]. Thus, in accordance with the accuracy of Eq. (2.5), we confine ourselves to the approximation

$$\tilde{M}_{2,n} \simeq \tilde{M}_{2,n}^0, \qquad \tilde{M}_{2,0}^0 = \tilde{M}_{2,2}^0 = a^3\delta_0/(3\pi), \qquad \tilde{M}_{2,1}^0 = 0. \tag{4.1}$$

Substitution of (4.1) into (3.2) yields

$$\Big(\frac{1}{c} + A_0 + \frac{a^2}{3}\,(C_{11} + C_{22})\Big)\tilde{F}_3 + \sum_{i=1}^{2} \tilde{M}_i A_0^{(i)} = \Big(1 - \frac{a^3}{3\pi}\,(A_0^{(2,0)} + A_0^{(2,2)})\Big)\delta_0.$$

Using similar reasoning to that used in deriving relation (4.1), we replace the last formula by the following one:

$$\Big(\frac{1}{c} + A_0 + \frac{a^2}{3}\,(C_{11} + C_{22}) + \frac{a^3}{3\pi c}(A_0^{(2,0)} + A_0^{(2,2)})\Big)\tilde{F}_3 + A_0^{(1)}\tilde{M}_1 + A_0^{(2)}\tilde{M}_2 = \delta_0. \tag{4.2}$$

Finally, Eq. (3.3) becomes

$$B_2\tilde{F}_3 + (1/m + B_2^{(1)})\tilde{M}_1 + B_2^{(2)}\tilde{M}_2 = \beta_1, \quad -B_1\tilde{F}_3 - B_1^{(1)}\tilde{M}_1 + (1/m - B_1^{(2)})\tilde{M}_2 = \beta_2. \tag{4.3}$$

Thus, the force F_3 and the moments M_1 and M_2 are related to the displacements of the punch δ_0 and its angles of rotation β_1 and β_2 by the approximate equations (4.2) and (4.3). Comparing (4.2) and (4.3) with (1.4), we obtain the asymptotic formulas for the normalized components of the local-compliance matrix $\tilde{\Pi}_{lk} = \pi E(1-\nu^2)^{-1}\Pi_{kl}$:

$$\tilde{\Pi}_{00} \simeq 1/c + A_0 + a^2(C_{11} + C_{22})/3 + a^2(A_0^{(2,0)} + A_0^{(2,2)})/6, \quad \tilde{\Pi}_{01} \simeq A_0^{(1)}, \quad \tilde{\Pi}_{02} \simeq A_0^{(2)},$$

$$\tilde{\Pi}_{10} \simeq B_2, \qquad \tilde{\Pi}_{11} \simeq 1/m + B_2^{(1)}, \qquad \tilde{\Pi}_{12} \simeq B_2^{(2)}, \tag{4.4}$$

$$\tilde{\Pi}_{20} \simeq -B_1, \qquad \tilde{\Pi}_{21} \simeq -B_1^{(1)}, \qquad \tilde{\Pi}_{22} \simeq 1/m - B_1^{(2)}.$$

Formulas (4.4) relate the components of the matrix Π to the capacity characteristics of the punch c and m (which depend only on the geometry of the punch base) and the coefficients in the asymptotic formula (2.3). The latter coefficients depend on the shape and size of the elastic body Ω, its fixing conditions, location of the point O, and Poisson's ratio.

Since the matrix Π is symmetric, we have

$$A_0^{(1)} = B_2, \qquad A_0^{(2)} = -B_1, \qquad B_1^{(1)} = -B_2^{(2)}. \tag{4.5}$$

The validity of these equalities can be verified by using the Betti formula and coefficients $A_0^{(1)}, \dots, B_2^{(2)}$ determined in [14, 17] as coefficients in asymptotic formulas of the type (2.3) for certain singular solutions. However, the reciprocity relations (4.5) and similar relations can easily be obtained directly from (2.3) with allowance for the following equality implied by the Betty theorem (see, e.g., [23, Chap. 4, § 3.1]):

$$g_3(y_1, y_2; x_1, x_2, 0) = g_3(x_1, x_2; y_1, y_2, 0). \tag{4.6}$$

Substituting expansion (2.3) into (4.6), we obtain (4.5) and the equalities

$$A_0^{(2,0)} = 2C_{11}, \qquad A_0^{(2,1)} = 2C_{12}, \qquad A_0^{(2,2)} = 2C_{22}. \tag{4.7}$$

Thus, in the general case, the reciprocity relations (4.5) and (4.7) reduce the number of different coefficients in the asymptotic solution constructed above.

Example 2. For a layer (see Example 1), formulas (2.4), (4.2), and (4.3) can be combined to give

$$\tilde{F}_3 = \frac{2a\delta_0}{\pi}\Big(1 - \frac{2a_0}{\pi}\,\varepsilon - \frac{8a_1}{3\pi}\,\varepsilon^3\Big)^{-1}, \qquad \tilde{M}_i = \frac{4a^3\beta_i}{3\pi}\Big(1 + \frac{8a_1}{3\pi}\,\varepsilon^3\Big)^{-1} \quad (i = 1, 2). \tag{4.8}$$

Expansion of expression (4.8) into a series in powers of the parameter $\varepsilon = a/h$ yields

$$\begin{gathered} F_3 = \frac{2E}{1-\nu^2}a\delta_0\Big\{1 + \frac{2a_0}{\pi}\,\varepsilon + \Big(\frac{2a_0}{\pi}\Big)^2\varepsilon^2 + \Big[\Big(\frac{2a_0}{\pi}\Big)^3 + \frac{8a_1}{3\pi}\Big]\varepsilon^3 + \Big[\Big(\frac{2a_0}{\pi}\Big)^4 + \frac{32a_0a_1}{3\pi^2}\Big]\varepsilon^4 + O(\varepsilon^5)\Big\}, \\ M_i = \frac{4E}{3(1-\nu^2)}\,a^3\beta_i\Big(1 - \frac{8a_1}{3\pi}\,\varepsilon^3 + O(\varepsilon^5)\Big) \quad (i = 1, 2). \end{gathered} \tag{4.9}$$

Formulas (4.9) coincide with formulas (48.2) and (50.2) in [1].

5. Calculation of the Coefficients A_0 and C_{11} for the Center of an Elastic Hemisphere. Let the elastic body Ω be shaped like a hemisphere of radius l and fixed over the spherical part of the boundary Γ_u. We use the Bubnov–Galerkin method to construct an approximate solution of the problem for a unit force applied to the center of the cut Σ.

Since the problem of determining the vector function $\boldsymbol{g}(0; \boldsymbol{x})$ [see (2.1)] is axisymmetric, one can use the general solution of the Lamé equations in cylindrical coordinates r and z, expressed in terms of two harmonic functions Φ_1 and Φ_2 in the Weber form (see [24, § 12]):

$$\begin{gathered} g_r = \frac{1-\nu^2}{\pi E}\,\frac{\partial}{\partial r}\Big[\Phi_1 + z\,\frac{\partial \Phi_2}{\partial z} + 2(1-\nu)\Phi_2\Big], \qquad g_z = \frac{1-\nu^2}{\pi E}\,\frac{\partial}{\partial z}\Big[\Phi_1 + z\,\frac{\partial \Phi_2}{\partial z} - 2(1-\nu)\Phi_2\Big], \\ \sigma_{zz} = \frac{1}{r}\,\frac{\partial}{\partial r}\Big(r\,\frac{\partial \Phi}{\partial r}\Big), \qquad \tau_{rz} = -\frac{\partial^2 \Phi}{\partial r \partial z}, \qquad \Phi = \Phi_1 + z\,\frac{\partial \Phi_2}{\partial z}. \end{gathered} \tag{5.1}$$

As stress functions, we use the homogeneous harmonic polynomials

$$\Phi_i^n = c_i^n \rho^n P_n(\cos\theta) \quad (i = 1, 2), \qquad \rho = \sqrt{r^2 + z^2}, \qquad \cos\theta = z/\sqrt{r^2 + z^2} \tag{5.2}$$

(P_n is the Legendre polynomial). Substituting expressions (5.1) and (5.2) into the conditions $\sigma_{zz} = 0$ and $\tau_{rz} = 0$ for $z = 0$, we eliminate one of the coefficients c_1^n or c_2^n: $c_1^n = 0$ for even n or $c_2^n = -c_1^n$ for odd n.

The homogeneous vector polynomial of the nth degree, which satisfies the homogeneous Lamé equations in the half-space $z > 0$ and the condition that the stresses vanish at the boundary $z = 0$, has the components

$$\begin{gathered} \tilde{V}_r^n = -(1-\alpha)(1/r)\Phi^{n+1} + [2 - (1+\alpha)(n+1)](z/r)\Phi^n + (1+\alpha)n(z^2/r)\Phi^{n-1}, \\ \tilde{V}_z^n = 2\Phi^n - (1+\alpha)nz\Phi^{n-1}, \qquad \alpha \equiv \nu(1-\nu)^{-1} \end{gathered} \tag{5.3}$$

for even n and

$$\begin{gathered} \tilde{V}_r^n = (2/r)\Phi^{n+1} + [(1+\alpha)n - 2](z/r)\Phi^n - (1+\alpha)n(z^2/r)\Phi^{n-1}, \\ \tilde{V}_z^n = -(1-\alpha)\Phi^n + (1+\alpha)nz\Phi^{n-1} \end{gathered} \tag{5.4}$$

for odd n. The first three vectors $\tilde{\boldsymbol{V}}^n$ calculated by formulas (5.3) and (5.4) have the form

$$\tilde{\boldsymbol{V}}^0 = \boldsymbol{e}_z, \qquad \tilde{\boldsymbol{V}}^1 = r\boldsymbol{e}_r - 2\alpha z\boldsymbol{e}_z, \qquad \tilde{\boldsymbol{V}}^2 = -2zr\boldsymbol{e}_r + (r^2 + 2\alpha z^2)\boldsymbol{e}_z.$$

TABLE 1

ν	A_0	$C_{11} = C_{22}$
0.2	−1.5442	0.5943
0.25	−1.6027	0.6642
0.3	−1.6899	0.7585
0.35	−1.8214	0.8891
0.4	−2.0236	1.0763

Thus, the cylindrical components of the regular part of the Green vector function $\boldsymbol{G}(0;\boldsymbol{x})$ satisfy the boundary conditions

$$\frac{\pi E}{1-\nu^2}\, g_r\Big|_{\Gamma_u} = -\frac{1}{2(1-\nu)}\,\frac{1}{l}\Big(\sin\theta\cos\theta - (1-2\nu)\,\frac{\sin\theta}{1+\cos\theta}\Big),$$
$$\frac{\pi E}{1-\nu^2}\, g_z\Big|_{\Gamma_u} = -\frac{1}{l}\Big(1 + \frac{1}{2(1-\nu)}\,\cos^2\theta\Big) \tag{5.5}$$

for $\rho = l$ and $0 \leqslant \theta \leqslant \pi/2$. We use the following approximation of the vector function $\boldsymbol{g}(0;\boldsymbol{x})$:

$$\boldsymbol{v}^N(r,z) = \frac{1-\nu^2}{\pi E}\sum_{n=0}^{N}\frac{c^n}{l^{n+1}}\,\tilde{\boldsymbol{V}}^n(r,z). \tag{5.6}$$

In this case, the unknown quantities are calculated by the formulas

$$A_0 = c^0/l, \qquad C_{11} = C_{22} = c^2/l^3.$$

To determine the coefficients c^0, c^1, …, and c^N, we obtain a system of $N+1$ linear algebraic equations using the condition that the discrepancy in the boundary conditions (5.5) due to approximation (5.6) is orthogonal to each vector $\tilde{\boldsymbol{V}}^0$, $\tilde{\boldsymbol{V}}^1$, …, and $\tilde{\boldsymbol{V}}^N$ over the hemisphere. The calculation results are listed in Table 1. The calculations were performed for $N = 2$–17. It should be noted that the relative error in determining A_0 is as small as 2% even for $N = 2$. To verify the results obtained, calculations by the method of boundary collocation for equidistant nodes were also performed.

Conclusions. To refine the asymptotic formula (2.3), it is necessary to use additional parameters that characterize the geometry of the elastic body. All local-compliance coefficients can be obtained by the complete asymptotic expansion. Practically, it is possible to obtain explicitly only a few first terms of the asymptotic representation of the contact pressure (see, e.g., [1, 5]).

By virtue of the reciprocity relations (4.5) and (4.7), expansion (2.3) is simplified:

$$(\pi E/(1-\nu^2))g_3(\boldsymbol{y};x_1,x_2,0) = A_0 + B_i x_i + B_i y_i + C_{ij}x_i x_j + b_{ij}x_i y_j + C_{ij}y_i y_j + \dots\ .$$

Here $b_{11} = -B_1^{(2)}$, $b_{12} = b_{21} = B_1^{(1)}$, and $b_{22} = B_2^{(1)}$; summation is performed over repeated indices. The condition that the linear and quadratic forms appearing in this expression are invariant with respect to rotation of the coordinate axes implies that, in passing to a new coordinate system, the quantities B_i and C_{ij} and b_{ij} should be transformed as a vector and tensors, respectively.

If the body Ω and the parts of the boundary Γ_u and Γ_σ are symmetric, then $b_{12} = 0$ and $b_{11} = b_{22}$ at the point O. However, the last coefficients are determined by solving the corresponding nonaxisymmetric problem.

Formulas (1.4) and (4.4) can be considered as an asymptotic model of an elastic body under point loads (forces and moments). It is well known (see, e.g., [25; 10, Chap. 10, § 1]) that the singular solution $\boldsymbol{G}(0,\boldsymbol{x})$ is meaningless in the neighborhood of the point at which the force is applied. In particular, the displacement at this point is unbounded, whereas formula (1.4) relates the force $F_3\boldsymbol{e}_3$ to the generalized displacement $\delta_0\boldsymbol{e}_3$ (compare with Example 1 in [26, § 8.9]).

It is also known (see [27, Chap. 7, § 21]) that the real distribution law of local loads is difficult to determine, whereas the resultant force vector is usually known with a high degree of accuracy. *A priori* knowledge of the distribution law of local loads is necessary (see [28, p. 301]) if, in calculations, they are reduced to point forces by passing to the limit (see [29] and [28, Chap. 3, § 6]). At the same time, to construct an asymptotic model of a point force, it is necessary to solve the coupled integral equation of the contact problem for an elastic body of finite dimensions and thereby to determine approximately the pressure under the base of a small punch.

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